AP Calculus Test Prep

Transcendental Functions

Vic Levine
Madison College
Madison, WI

I. Quick Review

An **algebraic** function of x is one that can be expressed as a finite number of sums, differences, multiples, quotients and radicals involving x^n. A function which is not algebraic is **transcendental**.

Algebraic: polynomials, rational expressions, radical expressions

Transcendental: trig, inverse trig, logarithmic and exponential

Trig Functions

You are expected to know the exact trig values for the basic reference angles: 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ in all quadrants.

Complete the following table without using a calculator:

Angle	sin x	cos x	tan x
0			
$\frac{\pi}{6}$			
$\frac{\pi}{4}$			
$\frac{\pi}{3}$			
$\frac{\pi}{2}$			

Limits and Derivatives

Most Important Trig Limits: $\lim_{x\to 0}\frac{\sin x}{x}=1$ and $\lim_{x\to 0}\frac{1-\cos x}{x}=0$

Trig Derivatives: Complete the following table:

f(x)	f'(x)
sin x	
cos x	
tan x	
cot x	
sec x	
csc x	

Chain Rule: The angle of the trig function never changes when doing a calculus operation!

Remember *l'Hôpital's Rule*: Let *f* and *g* be differentiable functions on an open interval (*a, b*) containing *c*, except perhaps at *c* itself. Assume that $g'(x) \neq 0$ for all *x* in (*a, b*), except perhaps at *c* itself. If $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0}$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$ provided the limit on the right exists (or is infinite). The result applies as well to any indeterminate of the form $\frac{\pm\infty}{\pm\infty}$.

Ex: Find $\lim_{x \to 0} \frac{\sin 3x}{\sin 4x}$

Integrals

Ex. Evaluate in exact form: $\int_0^{\pi/4} \cos 2x dx$

Ex: Find the exact area enclosed by the "triangular" region enclosed on the left by the *y*-axis and on the right by *f(x)* = sin *x* and *g(x)* = cos *x* in the first quadrant.

Ex: Set up, but do not evaluate the following definite integral problem:
Find the volume of the solid whose base is the region enclosed by the *y*-axis, the line *y* = 1 and *y* = tan *x*. Each cross-section perpendicular to the *x*-axis is a square

Logarithms and Exponentials

The natural logarithm of *x, ln x*, is defined to be $\int_1^x \frac{1}{t} dt$ for x > 0. Since this function is monotonically increasing on (0, ∞), it has an inverse which is a function. This function is the exponential function, e^x.

1) $\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$ and $\int \frac{1}{u} du = \ln|u| + C$

2) $\frac{d(e^u)}{dx} = e^u \frac{du}{dx}$ and $\int e^u du = e^u + C$

3) $\log_b x = \frac{\ln x}{\ln b}$

4) $\dfrac{d(a^u)}{dx} = a^u \ln a \dfrac{du}{dx}$ and $\int a^u du = \dfrac{1}{\ln a} a^u + C$

5) Exponential Growth: The rate of growth of a quantity is directly proportional to the size of the quantity.

$\dfrac{dy}{dt} = ky$ and $y(0)=y_0$

$y = y_0 e^{kt}$

Inverse Trig Functions

Derivative Rules: Let u be a differentiable function of *x*.

$$\frac{d}{dx}\left(\arcsin u\right) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\arctan u\right) = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\operatorname{arcsec} u\right) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

Integration Rules: Let u be a differentiable function of *x*, and let $a > 0$.

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$$

II. Sample Multiple Choice Questions

2008 AB4: $\int (\sin(2x)+\cos(2x))dx =$

(A) $\frac{1}{2}\cos(2x)+\frac{1}{2}\sin(2x)+C$ (B) $-\frac{1}{2}\cos(2x)+\frac{1}{2}\sin(2x)+C$ (C) $2\cos(2x)+2\sin(2x)+C$

(D) $2\cos(2x)-2\sin(2x)+C$ (E) $-2\cos(2x)+2\sin(2x)+C$

2008 AB8: If f(x) = cos (3x), then $f'\left(\frac{\pi}{9}\right)=$

(A) $\frac{3\sqrt{3}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $-\frac{\sqrt{3}}{2}$ (D) $-\frac{3}{2}$ (E) $-\frac{3\sqrt{3}}{2}$

2008 AB12: If $f(x)=e^{\left(\frac{2}{x}\right)}$, then f ’(x) =

(A) $2e^{\left(\frac{2}{x}\right)}\ln x$ (B) $e^{\left(\frac{2}{x}\right)}$ (C) $e^{\left(\frac{-2}{x^2}\right)}$ (D) $-\frac{2}{x^2}e^{\frac{2}{x}}$ (E) $-2x^2e^{\frac{2}{x}}$

2008 AB15: $\int \frac{x}{x^2-4}dx =$

(A) $\frac{-1}{4(x^2-4)^2}+C$ (B) $\frac{1}{2\left(x^2-4\right)}+C$ (C) $\frac{1}{2}\ln\left|x^2-4\right|+C$

(D) $2\ln\left|x^2-4\right|+C$ (E) $\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C$

2008 AB16: If sin(xy) = x, then $\frac{dy}{dx}=$

(A) $\frac{1}{\cos(xy)}$ (B) $\frac{1}{x\cos(xy)}$ (C) $\frac{1-\cos(xy)}{\cos(xy)}$

(D) $\frac{1-y\cos(xy)}{x\cos(xy)}$ (E) $\frac{y(1-\cos(xy))}{x}$

2008 AB19: What are horizontal asymptotes of the graph of $y=\frac{5+2^x}{1-2^x}$ in the xy-plane?

(A) y = -1 only (B) y = 0 only (C) y = 5 only

(D) y = -1 and y = 0 (E) y = -1 and y = 5

2008 AB22: A rumor spreads among a population of N people at a rate proportional to the product of the number if people who have heard the rumor and the number of people who have not heard the rumor. If *p* denotes the number of people who have heard the rumor, which of the following differential equations could be used to model this situation with respect to time, *t*, where *k* is a positive constant?

(A) $\frac{dp}{dt} = kp$ (B) $\frac{dp}{dt} = kp(N-p)$ (C) $\frac{dp}{dt} = kp(p-N)$ (D) $\frac{dp}{dt} = kt(N-t)$ (E) $\frac{dp}{dt} = kt(t-N)$

2008 AB26: What is the slope of the line tangent to the curve y = arctan (4x) at the point where x = $\frac{1}{4}$?

(A) 2 (B) $\frac{1}{2}$ (C) 0 (D) $-\frac{1}{2}$ (E) – 2

2008 AB78: The first derivative of the function *f* is defined by $f'(x) = \sin\left(x^3 - x\right)$ for $0 \le x \le 2$. On what intervals is f increasing?

(A) $1 \le x \le 1.445$ only (B) $1 \le x \le 1.691$ (C) $1.445 \le x \le 1.875$

(D) $0.577 \le x \le 1.445$ and $1.875 \le x \le 2$ (E) $0 \le x \le 1$ and $1.691 \le x \le 2$

2008 AB80: The derivative of the function *f* is given by $f'(x) = x^2\cos\left(x^2\right)$. How many points of inflection does the graph of *f* have on the open interval (-2, 2)?

(A) One (B) Two (C) Three (D) Four (E) Five

2008 AB82: A particle moves along a straight line with velocity given by $v(t) = 7 - (1.01)^{-t^2}$ at time $t \ge 0$. What is the acceleration of the particle at time *t* = 3?

(A) -0.914 (B) 0.055 (C) 5.486 (D) 6.086 (E) 18.087

2008 AB91: What is the average value of $y = \frac{\cos x}{x^2 + x + 2}$ on the interval ‘[-1, 3]?

(A) -0.085 (B) 0.090 (C) 0.183 (D) 0.244 (E) 0.732

III. Sample Free Response Questions

2008 AB6: Let *f* be the function given by $f(x)=\frac{\ln x}{x}$ for all x > 0. The derivative of *f* is given by $f'(x)=\frac{1-\ln x}{x^2}$.

(a) Write an equation for the line tangent to the graph of *f* at x = e^2.

(b) Find the x-coordinate of the critical point of *f*. Determine whether this point is relative minimum, relative maximum, or neither for the function *f*. Justify your answer.

(c) The graph of the function *f* has exactly one point of inflection. Find the x-coordinate of this point.

(d) Find $\lim_{x\to o^+} f(x)$.

2009B AB1: At a certain height, a tree trunk has a circular cross section. The radius *R(t)* of that cross section grows at a rate modeled by the function $\frac{dR}{dt}=\frac{1}{16}\left(3+\sin(t^2)\right)$ centimeters per year for $0 \le t \le 3$, where *t* is measured in years. At time *t* = 0, the radius is 6 centimeters. The area of the cross section at time *t* is denoted by *A(t)*.

(a) Write an expression, involving an integral, for the radius *R(t)* for $0 \le t \le 3$. Use your expression to find *R*(3).

(b) Find the rate at which the cross-sectional area *A(t)* is increasing at time *t* = 3 years. Indicate units of measure.

(c) Evaluate $\int_0^3 A'(t)dt$. Using appropriate units, interpret the meaning of that integral in terms of cross-sectional area.

2009B AB2: A storm washed away sand from a beach, causing the edge of the water to get closer to a nearby road. The rate at which the distance between the road and the edge of the water was changing during the storm is modeled by $f(t) = \sqrt{t} + \cos t - 3$ meters per hour, t hours after the storm began. The edge of the water was 35 meters from the road when the storm began, and the storm lasted 5 hours. The derivative of *f(t)* is $f'(t) = \frac{1}{2\sqrt{t}} - \sin t$.

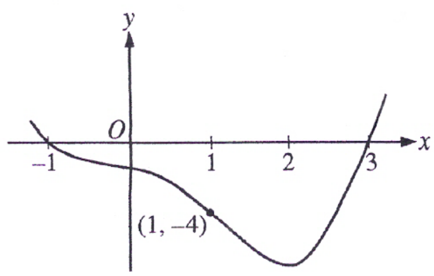
(a) What was the distance between the road and the edge of the water at the end of the storm?

(b) Using correct units, interpret the value $f'(4)=1.007$ in terms of the distance between the road and the edge of the water.

(c) At what time during the 5 hours of the storm was the distance between the road and the edge of the water decreasing most rapidly? Justify your answer.

(d) After the storm, a machine pumped sand back onto the beach so that the distance between the road and the edge of the water was growing at a rate of *g(p)* meters per day, where *p* is the number of days since pumping began. Write an equation involving an integral expression whose solution would give the number of days that sand must be pumped to restore the original distance between the road and the edge of the water.

2009B AB5:



Graph of f'

Let f be a twice-differentiable function defined on the interval $-1.2 < x < 3.2$ with $f(1) = 2$. The graph of f', the derivative of f, is shown above. The graph of f' crosses the x-axis at $x = -1$ and $x = 3$ and has a horizontal tangent at $x = 2$. Let g be the function given by $g(x) = e^{f(x)}$.

(a) Write an equation for the line tangent to the graph of g at $x = 1$.

(b) For $-1.2 < x < 3.2$, find all values of x at which g has a local maximum. Justify your answer.

(c) The second derivative of g is $g''(x) = e^{f(x)}\left[(f'(x))^2 + f''(x)\right]$. Is $g''(-1)$ positive, negative or zero? Justify your answer.

(d) Find the average rate of change of g', the derivative of g, over the interval [1, 3].